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Tollmien–Schlichting wave/Dean vortex interactions in curved channel flow

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Evidence from direct numerical simulations and from a recent weakly nonlinear theory is presented which shows that the weakly nonlinear results of Daudpota, Hall & Zang (1988) make incorrect predictions for the influence of the Tollmien–Schlichting wave on the Dean vortex.

1. Introduction

Very mildly curved channel flows can support both unstable Tollmien–Schlichting (TS) waves and Dean vortices. Daudpota, Hall & Zang (1988) (hereinafter referred to as DHZ) developed a weakly nonlinear interaction theory to study the interaction of TS waves with Dean vortices at finite Reynolds number. They employed a multiple scale version of the Stuart (1960) and Watson (1960) approach to derive two coupled Landau equations for the perturbation amplitudes of the Dean vortices and TS waves. Here we present evidence that shows that their results are in error with respect to the influence of the TS wave on the Dean vortex. In §2, we enumerate the nomenclature and introduce the coupled Landau equations which describe the evolution of the disturbances. In §3, we illustrate some discrepancies between the results of direct numerical simulations (DNS) and the predictions of DHZ, and in §4, we present the results of a new weakly nonlinear theory and show that its results are consistent with the DNS.

2. Basic nomenclature

The incompressible flow in a mildly curved channel is driven by an azimuthal pressure gradient. The flow is assumed to be periodic in the axial and azimuthal directions; any disturbances evolve in time. The radial, or wall-normal direction, extends from an inner radius, r_i, to an outer radius, r_o. Here we non-dimensionalize all spatial coordinates with the channel half-width h, velocities with the bulk velocity $\bar{U}$, and pressure with $\rho\bar{U}^2$, where ρ is the density. The temporal scale is $h/\bar{U}$. The Reynolds number is defined as

$$Re = \frac{\bar{U}h}{\nu},$$

where ν is the kinematic viscosity. A curvature parameter, $\lambda = 1 - r_i/r_o$, is close to zero for the mildly curved channels considered here. Note that DHZ non-dimensionalize differently but their results can be easily converted into this system. The linear TS waves are assumed to be proportional to $\exp \mathrm{i}\aleph\theta$ where θ is the azimuthal coordinate and $\aleph$ is the azimuthal wavenumber. The linear Dean vortices are proportional to $\exp \mathrm{i}\beta z$, where z is the axial coordinate and β is axial wavenumber. The nonlinear structure of these disturbances also contain their harmonics.

DHZ use the solutions of the linearized equations as forcing terms in the weakly nonlinear expansions of the TS waves and the Dean vortices. In a manner similar to Watson (1960), they use solvability conditions at third order in the amplitude to determine the coefficients of a set of coupled Landau equations truncated at third order. Letting A represent an amplitude of the TS disturbance and B an amplitude of the Dean disturbance, one can write the real equations:

$$\dot{A}/A = a_{0,0}+a_{1,0}A^2+a_{0,1}B^2, \tag{1}$$

$$\dot{B}/B = b_{0,0}+b_{1,0}A^2+b_{0,1}B^2. \tag{2}$$

Here $a_{0,0}$ and $b_{0,0}$ are the linear growth rates of the TS and Dean disturbances, respectively. The coefficients $a_{1,0}$ and $b_{0,1}$ govern the self-interaction of the disturbances with themselves. The coefficients $a_{0,1}$ and $b_{1,0}$ determine the effect that one disturbance has on the other.

3. Comparison with DNS

Equations (1) and (2) admit four possible steady-state solutions: the trivial solution, finite A with $B = 0$, finite B with $A = 0$, and a combined state with finite values of both A and B. The behaviour of the solutions in the vicinity of these equilibrium points is easily checked via DNS.

Singer & Zang (1989) used a numerical simulation code which employed a curved channel variant of the method described by Zang & Hussaini (1986) with the nonlinear terms in skew-symmetric form (Zang 1991). They extensively studied the situation where $Re = 6291.67$, $\lambda = 2.189\times 10^{-5}$, $\aleph = 74257$, and $\beta = 4.508$. The constants in (1) and (2) obtained from DHZ are given in table 1. The linear growth rates, $a_{0,0}$ and $b_{0,0}$, are positive while the self-interaction coefficients, $a_{1,0}$ and $b_{0,1}$, are negative. This implies that either disturbance alone in the flow will evolve towards an equilibrium state. The negative value of $a_{0,1}$ indicates that the Dean vortex tends to stabilize the TS wave, while the positive value of $b_{1,0}$ means that the presence of the TS wave tends to destabilize the Dean vortex. Clearly, if both types of disturbance are in the flow initially, the theory predicts that the TS wave will decay, while the Dean vortex will ultimately go to its equilibrium state.

Four different types of initial conditions were used in the DNS of Singer & Zang (1989). In one case, the initial conditions included only a TS wave. An approximately periodic state was reached with an amplitude that differed from that predicted by the theory by approximately 15%. A second case included only a Dean vortex. An equilibrium state was reached with an amplitude that was 5% different than that predicted by the theory. These two simulations suggested that the self-interaction coefficients are essentially correct. In order to check the other interaction coefficients, both types of disturbances were included in the initial conditions. Starting a calculation with the amplitude of the Dean vortex approximately at its equilibrium level and the amplitude of the TS wave at approximately $\frac{1}{12}$ its predicted equilibrium value, the TS wave decayed and the Dean vortex evolved towards its equilibrium state. This was in accordance with the theory. However, when the amplitude of the TS wave was increased to $\frac{5}{6}$ its predicted equilibrium value, the Dean vortex decayed and the TS wave evolved towards its equilibrium value. This

	$a_{0,0}$	$a_{1,0}$	$a_{0,1}$	$b_{0,0}$	$b_{1,0}$	$b_{0,1}$
DHZ	1.20×10^{-4}	-8.85	-101000	3.97×10^{-5}	1.23×10^{5}	-8470
SEZ	1.21×10^{-4}	-8.25	-95000	3.97×10^{-5}	-482	-7640

TABLE 1. Comparison of Landau coefficients for $Re = 6291.67$, $\lambda = 2.189\times10^{-5}$, $\aleph = 74257$, and $\beta = 4.508$

was in direct conflict with the theory of DHZ. Another simulation was performed with the same initial conditions, but the Dean vortex was artificially maintained at its original amplitude. In this case, the TS wave again decayed. This suggested that while $a_{0,1}$, the coefficient that controls the effect of the Dean vortex on the TS wave, has at least the proper sign, $b_{1,0}$, the coefficient that governs the effect of the TS wave on the Dean vortex, is wrong. Additional simulations with different parameters also supported this conjecture.

4. Comparison with a new theory

Recently Singer, Erlebacher & Zang (1992) reformulated the weakly nonlinear theory for curved channel flow using the perturbation approach of Herbert (1980, 1983). The results of this theory were then used to resolve the discrepancy between DHZ and Singer & Zang (1989). As seen in table 1, the theories of Singer *et al.* (1992) and DHZ produced similar values for all the coefficients except $b_{1,0}$, the coefficient which was considered suspect in the previous section. Another case with different flow parameters also isolated $b_{1,0}$ as the only coefficient which had a significantly different value. The results of the theory of Singer *et al.* (1992) were checked extensively with direct numerical simulations and in all cases there was at least qualitative agreement.

5. Conclusions

An extensive comparison of the results of the weakly nonlinear theory of DHZ with both DNS and the weakly nonlinear theory of Singer *et al.* (1992) indicates that there is an error in the results of DHZ. This error manifests itself in the coefficient $b_{1,0}$, which governs the effect of a small-amplitude TS wave on a Dean vortex. Even the qualitative behaviour of the disturbances can be significantly altered by this error, hence we recommend that the results of DHZ not be used to verify other theories as was suggested by Bennett, Hall & Smith (1991).

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